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RATES OF CONVERGENCE FOR RANDOM APPROXIMATIONS OF CONVEX SETS

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Abstract

The Hausdorff distance between a compact convex set $K \subset \mathbb{R}^d$ and random sets $\hat{K} \subset \mathbb{R}^d$ is studied. Basic inequalities are derived for the case of $\hat{K}$ being a convex subset of K. If applied to special sequences of such random sets, these inequalities yield rates of almost sure convergence. With the help of duality considerations these results are extended to the case of $\hat{K}$ being the intersection of a random family of halfspaces containing K.

CONVEX SET; DUALITY; PACKING NUMBERS; POLAR SET; PROBABILITY INEQUALITY; RATE OF CONVERGENCE

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1. Introduction

Let K be a convex compact set in $\mathbb{R}^d$ with non-empty interior. The following two types of problem have been studied extensively.

I: How closely is K approximated by the convex hull $\operatorname{conv}(Z)$ of a random subset Z of K?
II: How closely is K approximated by the intersection $\bigcap_{H \in \mathscr{H}} H$ of a random family $\mathscr{H}$ of halfspaces containing K?

The reader is referred to the very informative reviews of Schneider (1988) and Weil and Wieacker (1993). A typical example for Z in Problem I is $\{X_1, X_2, \cdots, X_n\}$ with independent, identically distributed random points $X_1, X_2, X_3, \cdots \in K$. Problem II arises in the reconstruction of K from projections onto random lower-dimensional subspaces, e.g. Small (1991). Numerous results for both problems have been obtained under the special assumption that K either is a polytope or has sufficiently smooth boundary ∂K. In many cases the approximation error is measured by Lebesgue measure of the symmetric difference $K \Delta \hat{K}$, where $\hat{K}$ stands for $\operatorname{conv}(Z)$ or $\bigcap_{H \in \mathscr{H}} H$. Another interesting quantity is the Hausdorff distance $d_{\mathrm{H}}(K, \hat{K})$, but

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this is technically more difficult and most known results are for dimension $d = 2$ only. An exception is Barany (1989); see also Section 2.

In what follows we investigate the Hausdorff distance $d_H(\hat{K}, K)$ in an arbitrary dimension d and for arbitrary K, i.e. ∂K is allowed to have vertices and flat spots. Theorem 1 in Section 2 gives basic inequalities for Problem I. These inequalities are applied to various special sequences of random sets. For instance, let the random points X_i be uniformly distributed on ∂K. Then with probability one,

$$d_H(K, \operatorname{conv}(\{X_1, X_2, \cdots, X_n\})) = \begin{cases} O((\log n/n)^{1/(d-1)}) & \text{in general,} \\ O((\log n/n)^{2/(d-1)}) & \text{if } \partial K \text{ is smooth.} \end{cases}$$

With the help of duality considerations it is shown in Section 3 that Problems I and II are equivalent in some sense. The results of Section 2 are then extended to Problem II. For example, if K is reconstructed from its orthographic shadows in n independent, uniformly distributed directions, the approximation error is of order $O(\log n/n)$ in general and $O((\log n/n)^2)$ if ∂K is strictly curved.

All proofs are deferred to Section 4.

2. Various results for Problem I

Let $\mathbb{R}^d$ be equipped with the standard inner product $\langle \cdot, \cdot \rangle$ and Euclidean norm $|\cdot|$. For $x \in \mathbb{R}^d$ and $M \subset \mathbb{R}^d$ define $d(x, M) := \inf_{m \in M} |x - m|$. Further let $B(M, \varepsilon) := \{y \in \mathbb{R}^d : d(y, M) \leqq \varepsilon\}$ and $B(x, \varepsilon) := B(\{x\}, \varepsilon)$. Then the Hausdorff distance between M and $L \subset \mathbb{R}^d$ is defined as

$$d_H(L, M) := \inf\{\varepsilon > 0 : M \subset B(L, \varepsilon) \text{ and } L \subset B(M, \varepsilon)\}.$$

There are some elementary inequalities for $d_H(K, \operatorname{conv}(Z))$ in terms of the function $\pi(K, Z, \varepsilon) := \sup_{x \in \partial K} \boldsymbol{P}\{B(x, \varepsilon) \cap Z = \varnothing\}$ and the packing numbers $D(K, \varepsilon) := \max\{\#(F) : F \subset \partial K, |x - y| > \varepsilon \text{ for different } x,y \in F\}$ of ∂K. ($\#(.)$ denotes cardinality.) Covering ∂K with closed balls is more convenient than the 'cap coverings' used by Barany (1989) and allows for similar results with simple additional considerations. We call ∂K 'smooth' if the following condition is satisfied.

Condition (S). *For each $x \in \partial K$ there is a unique $\theta(x) \in \partial B(0, 1)$ such that $\langle y, \theta(x) \rangle \leqq \langle x, \theta(x) \rangle$ for all $y \in K$, and for some constant $l \in \mathbb{R}$, $|\theta(x) - \theta(y)| \leqq l\,|x - y|\ \forall x,y \in \partial K$.*

Theorem 1 [a]. *For arbitrary $\varepsilon > 0$, $\boldsymbol{P}\{\partial K \not\subset B(Z, 2\varepsilon)\} \leqq D(K, \varepsilon)\pi(K, Z, \varepsilon)$.*
[b] *For $\delta > 0$ let Z_δ be any subset of K such that $\partial K \subset B(Z_\delta, \delta)$. Then*

$$d_H(K, \operatorname{conv}(Z_\delta)) \leqq \begin{cases} \delta & \textit{in general,} \\ l\delta^2 & \textit{if (S) holds and } Z_\delta \subset \partial K. \end{cases}$$

An elementary bound for $D(K, \varepsilon)$, which is sufficient for our purposes, is given by

$$(1) \qquad D(K, \varepsilon) \leqq (2 \text{ diameter } (K)\varepsilon^{-1} + 1)^d \quad \forall \varepsilon > 0;$$

see Pollard (1990, Section 4). The main problem will be to find good bounds for $\pi(K, Z, \varepsilon)$.

First we consider $Z_{n,1} := \{X_1, X_2, \cdots, X_n\}$ with independent, identically distributed random points $X_1, X_2, X_3, \cdots \in K$. For a sequence of real random variables $(Y_n)_n$ and a sequence of positive numbers $(c_n)_n$ we say here that $Y_n = O(c_n)$ almost surely if $\limsup_{n\to\infty} |c_n^{-1} Y_n| \leqq c$ almost surely for some fixed $c < \infty$.

Corollary 1. *Let X_i be uniformly distributed on K, i.e. $\boldsymbol{P}\{X_i \in A\} =$ Leb $(A)/$Leb (K) for any Borel set $A \subset K$, where* Leb *denotes Lebesgue measure on $\mathbb{R}^d$. Then*

$$d_{\mathrm{H}}(K, \operatorname{conv}(Z_{n,1})) = \begin{cases} O((\log n/n)^{1/d}) & \textit{almost surely,} \\ O((\log n/n)^{2/(d+1)}) & \textit{almost surely under } (S). \end{cases}$$

Under stronger regularity conditions on ∂K, Barany (1989, Theorem 6) shows that the expectation of $d_{\mathrm{H}}(K, \operatorname{conv}(Z_{n,1}))$ is precisely of order $(\log n/n)^{2/(d+1)}$.

Corollary 2. *Let X_i be uniformly distributed on ∂K, i.e. $\boldsymbol{P}\{X_i \in A\} =$ $\mathrm{H}_{d-1}(A)/\mathrm{H}_{d-1}(\partial K)$ for any Borel set $A \subset \partial K$, where H_{d-1} denotes $(d-1)$-dimensional Hausdorff measure on $\mathbb{R}^d$. Then*

$$d_{\mathrm{H}}(K, \operatorname{conv}(Z_{n,1})) = \begin{cases} O((\log n/n)^{1/(d-1)}) & \textit{almost surely,} \\ O((\log n/n)^{2/(d-1)}) & \textit{almost surely under } (S). \end{cases}$$

These results remain valid if the given distribution P of X_i is replaced with another distribution $Q \geqq aP$ for some $a > 0$.

In a different application, which is motivated by ultrasound pictures, one considers intersections of K with independent, identically distributed hyperplanes. To make this precise, we assume from now on that 0 is an interior point of K and define $\rho = \rho(K) := \max\{r > 0 : B(0, r) \subset K\}$, $R = R(K) := \min\{r > 0 : B(0, r) \supset K\}$. Now let $U_1, U_2, U_3, \cdots$ and $A_1, A_2, A_3, \cdots$ be independent random variables, where U_i is uniformly distributed on $\partial B(0, 1)$ and A_i is uniformly distributed on $[-R, R]$. Then define $Z_{n,2} := \cup_{1\leqq i\leqq n} \{x \in K : \langle x, U_i\rangle = A_i\}$.

Corollary 3.

$$d_{\mathrm{H}}(K, \operatorname{conv}(Z_{n,2})) = \begin{cases} O(\log n/n) & \textit{almost surely,} \\ O((\log n/n)^2 & \textit{almost surely under } (S). \end{cases}$$

3. Equivalence of Problems I and II

The equivalence of Problems I and II can be stated in terms of the *polar set* $M^* := \{y \in \mathbb{R}^d : \langle x, y\rangle \leqq 1 \text{ for all } x \in M\}$ of $M \subset \mathbb{R}^d$. It follows from this definition that M^* is a closed, convex set such that $0 \in M^* = \operatorname{conv}(M \cup \{0\})^*$. In particular, since $0 \in \operatorname{int}(K)$,

$$(2) \qquad K^* \text{ is compact}, \quad 0 \in \operatorname{int}(K^*), \quad \text{and } (K^*)^* = K;$$

see Schneider (1993, Theorem 1.6.1). In fact, it will be convenient to consider Problem II with K^* in place of K. Beforehand we define the mapping $\mathbb{R}^d \setminus \{0\} \ni x \mapsto \phi(x) := |x|^{-1}x$, the support function $\partial B(0, 1) \ni s \mapsto h(M, s) := \sup_{x \in M}\langle x, s\rangle$ of M, which coincides with $h(\operatorname{conv}(M), .)$, and the supporting halfspaces $H(M, s) := \{x \in \mathbb{R}^d : \langle x, s\rangle \leqq h(M, s)\}$.

Theorem 2. *Let S be a random, closed subset of $\partial B(0, 1)$, and define $Z := \phi^{-1}(S) \cap \partial K$. Then the following inequalities hold*:

$$R^{-2}d_{\mathrm{H}}(K, \operatorname{conv}(Z)) \leqq d_{\mathrm{H}}\left(K^*, \bigcap_{s \in S} H(K^*, s)\right) \leqq \frac{\rho^{-2}d_{\mathrm{H}}(K, \operatorname{conv}(Z))}{(1 - \rho^{-1}d_{\mathrm{H}}(K, \operatorname{conv}(Z)))^+},$$

$$\pi(K, Z, \rho^{-1}R^2\varepsilon) \leqq \pi(B(0, 1), S, \varepsilon) \leqq \pi(K, Z, \rho\varepsilon) \quad \forall \varepsilon > 0.$$

With Theorem 2 at hand one can extend results from Section 2 to Problem II, thus obtaining rates of convergence for the reconstruction of K^* from random projections onto one-dimensional subspaces and onto $(d-1)$-dimensional subspaces, as in Small (1991). The dual property to smoothness of ∂K is $\partial(K^*)$ having 'no flat parts':

Condition (NF). *For some constant* $l^* > 0$, $\min_{s \in \Theta(K^*, x), t \in \Theta(K^*, y)} |s - t| \geqq l^* |x - y| \ \forall x, y \in \partial(K^*)$, *where* $\Theta(K^*, x) := \{s \in \partial B(0, 1) := \langle x, s\rangle = h(K^*, s)\}$.

Corollary 4. *For independent, uniformly distributed* $U_1, U_2, U_3, \cdots \in \partial B(0, 1)$ *let* $S_{n,1} := \{U_1, U_2, \cdots, U_n\}$, *and* $S_{n,2} := \cup_{1 \leqq i \leqq n} \{U_i\}^{\perp} \cap \partial B(0, 1)$, *where* $(.)^{\perp}$ *denotes the orthogonal complement. Then*

$$d_{\mathrm{H}}\left(K^*, \bigcap_{s \in S_{n,1}} H(K^*, s)\right) = \begin{cases} O((\log n/n)^{1/(d-1)}) & \textit{almost surely}, \\ O((\log n/n)^{2/(d-1)}) & \textit{almost surely under (NF)}, \end{cases}$$

$$d_{\mathrm{H}}\left(K^*, \bigcap_{s \in S_{n,2}} H(K^*, s)\right) = \begin{cases} O(\log n/n) & \textit{almost surely}, \\ O((\log n/n)^2) & \textit{almost surely under (NF)}. \end{cases}$$

4. Proofs

Proof of Theorem 1. Let F be a maximal subset of ∂K such that $|x - y| > \varepsilon$ for different $x, y \in F$. Then $\partial K \subset B(F, \varepsilon)$, $\#(F) \leqq D(K, \varepsilon)$, and part [a] follows from

$$\begin{aligned} \boldsymbol{P}\{\partial K \not\subset B(Z, 2\varepsilon)\} &\leqq \boldsymbol{P}\{F \not\subset B(Z, \varepsilon)\} \\ &\leqq \sum_{x \in F} \boldsymbol{P}\{Z \cap B(X, \varepsilon) = \varnothing\} \\ &\leqq D(K, \varepsilon)\pi(K, Z, \varepsilon). \end{aligned}$$

Since $\operatorname{conv}(Z_\delta)$ is contained in K and convex, the general inequality in part [b] follows from $d_{\mathrm{H}}(K, \operatorname{conv}(Z_\delta)) = \sup_{x\in K} d(x, \operatorname{conv}(Z_\delta)) = \sup_{x\in\partial K} d(x, \operatorname{conv}(Z_\delta)) \leqq \sup_{x\in\partial K} d(x, Z_\delta) \leqq \delta$. If condition (S) holds and $Z_\delta \subset \partial K$, we utilize the following equality for arbitrary convex sets $L, M \subset \mathbb{R}^d$ such that at least one of them is bounded:

$$(3) \qquad d_{\mathrm{H}}(L, M) = \|h(L, \cdot) - h(M, \cdot)\| := \max_{s\in\partial B(0,1)} |h(L, s) - h(M, s)|;$$

see, for instance, Schneider (1993, Theorem 1.8.11). Here this equality leads to $d_{\mathrm{H}}(K, \operatorname{conv}(Z_\delta)) = \|h(K, \cdot) - h(Z_\delta, .)\| = \max_{s\in\partial B(0,1)} (h(K, s) - h(Z_\delta, s))$. Let the latter maximum be attained in $s_0 \in \partial B(0, 1)$. Further let $x_0 \in \partial K$ such that $h(K, s_0) = \langle x_0, s_0\rangle$. Then $s_0 = \theta(x_0)$, and $d_{\mathrm{H}}(K, \operatorname{conv}((Z_\delta)) = \inf_{z\in Z_\delta} \langle x_0 - z, \theta(x_0)\rangle = \min_{z\in \operatorname{cl}(Z_\delta)} \langle x_0 - z, \theta(x_0)\rangle$, where $\operatorname{cl}(\cdot)$ denotes closure. However, by assumption, $\langle x_0 - z, \theta(x_0)\rangle \leqq \langle x_0 - z, \theta(x_0) - \theta(z)\rangle \leqq l\,|x_0 - z|^2 \leqq l\delta^2$ for some $z \in \operatorname{cl}(Z_\delta)$.

The following lemma is of general use later for deducing rates of convergence.

Lemma 1. *Let $Z_n = \bigcup_{i=1}^n C_i$ with independent, identically distributed random subsets $C_1, C_2, C_3, \cdots$ of K. Suppose that*

$$(4) \qquad \inf_{x\in\partial K} \boldsymbol{P}\{C_1 \cap B(x, \varepsilon) \neq \emptyset\} \geqq \alpha\varepsilon^\beta \qquad \forall \varepsilon \in [0, \gamma]$$

with constants $\alpha, \beta, \gamma > 0$. Then $\pi(K, Z_n, \varepsilon) \leqq \exp(-n\alpha\varepsilon^\beta)$ for all $\varepsilon \in [0, \gamma]$, and

$$d_{\mathrm{H}}(K, \operatorname{conv}(Z_n)) = \begin{cases} O((\log n/n)^{1/\beta}) & \textit{almost surely,} \\ O((\log n/n)^{2/\beta}) & \textit{almost surely under } (S) \textit{ if } \boldsymbol{P}\{Z_n \subset \partial K\} = 1. \end{cases}$$

Proof of Lemma 1. Note first that $\pi(K, Z_n, \varepsilon)$ equals

$$(1 - \inf_{x\in\partial K} \boldsymbol{P}\{C_1 \cap B(x, \varepsilon) \neq \emptyset\})^n \leqq (1 - \alpha\varepsilon^\beta)^n \leqq \exp(-n\alpha\varepsilon_n^\beta)$$

for arbitrary $\varepsilon \in [0, \gamma]$. Now let $\varepsilon_n := \min\{\alpha, (c\log n/n)^{1/\beta}\}$. Then Theorem 1[a] and (1) show that $\sum_{n=1}^\infty \boldsymbol{P}\{\partial K \not\subset B(Z_n, 2\varepsilon_n)\} \leqq (4R + \alpha)^d \sum_{n=1}^\infty \varepsilon_n^{-d} \exp(-\alpha n\varepsilon_n^\beta) < \infty$ if c is sufficiently large. Hence the assertion follows from the Borel–Cantelli lemma together with Theorem 1[b].

Proof of Corollary 1. In order to prove the assertion for general K, it suffices to show that (4) holds with $C_i = \{X_i\}$ and $\beta = d$. In other words, one needs a reasonable lower bound for $\operatorname{Leb}(B(x, \varepsilon) \cap K)$, $x \in \partial K$. For that purpose let $s \in \partial B(0, 1)$ such that $\langle x, s\rangle \leqq -|x|\,(1 - (\rho/R)^2/2)$. Then $|x + |x|\,s|^2 = 2\,|x|^2 + 2\,|x|\,\langle x, s\rangle \leqq \rho^2$. Since $\operatorname{conv}(\{x\} \cup B(0, \rho)) \subset K$ and $|x| \geqq \rho$, this implies that $x + \{y \in B(0, \varepsilon) : \langle x, y\rangle \leqq -|x|\,|y|\,(1 - (\rho/R)^2/2)\}$ is a subset of $B(x, \varepsilon) \cap K$ for any $\varepsilon \in [0, \rho]$. Its Lebesgue measure equals $\tilde{\alpha}\varepsilon^d$, where $\tilde{\alpha} := \operatorname{Leb}\{y \in B(0, 1) : y_1 \geqq |y|\,(1 - (\rho/R)^2/2)\}$.

In the case of smooth ∂K one cannot apply Lemma 1 directly, because $Z_{n,1}$ is not a subset of ∂K. Alternatively let $\varepsilon_n := (c\log n/n)^{1/(d+1)}$ for some $c > 0$ and define $Z_n := Z_{n,1} \cap B(\partial K, \varepsilon_n^2)$. Then obviously $d_{\mathrm{H}}(K, \operatorname{conv}(Z_{n,1})) \leqq d_{\mathrm{H}}(K, \operatorname{conv}(Z_n))$, and the subset $Z'_n := B(Z_n, \varepsilon_n^2) \cap \partial K \subset \partial K$ satisfies $\varepsilon_n^2 \geqq d_{\mathrm{H}}(Z_n, Z'_n) \geqq d_{\mathrm{H}}(\mathrm{conv}\,(Z_n), \mathrm{conv}\,(Z'_n))$. Suppose that $\partial K \subset B(Z_n, 2\varepsilon_n)$. Then $\partial K \subset B(Z'_n, 2\varepsilon_n + \varepsilon_n^2)$, and Theorem 1[b] gives $d_{\mathrm{H}}(K, \mathrm{conv}\,(Z_n)) \leqq d_{\mathrm{H}}(K, \mathrm{conv}\,(Z'_n)) + \varepsilon_n^2 \leqq \varepsilon_n^2 + l(2\varepsilon_n + \varepsilon_n^2)^2$.

Thus, by Theorem 1[a] and (1), it suffices to show that $\sum_{n=1}^{\infty} \varepsilon_n^{-d} \pi(K, Z_n, \varepsilon_n) < \infty$ for suitable $c > 0$. This is certainly true if

$$\text{(5)} \qquad \mathrm{Leb}\,(K \cap B(x, \varepsilon) \cap B(\partial K, \varepsilon^2)) \geqq \alpha' \varepsilon^{d+1} \qquad \forall \varepsilon \in [0, \gamma]\, \forall x \in \partial K$$

with constants $\alpha', \gamma > 0$. We only sketch a proof of (5). For arbitrary $x, y \in \partial K$ condition (S) implies that $\langle x - y, \theta(x) \rangle \leqq l\,|x - y|^2$. With $\tilde{\rho} := (2l)^{-1}$ this can be rewritten as $|y - (x - \tilde{\rho}\theta(x))| \geqq \tilde{\rho}$. This implies that $B(x - \tilde{\rho}\theta(x), \tilde{\rho}) \subset K$. Moreover one easily verifies that $K \cap B(\partial K, \varepsilon^2)$ contains the set $\{z \in K : \langle x - z, \theta(x) \rangle \leqq \varepsilon^2\}$. Consequently $\mathrm{Leb}\,(K \cap B(x, \varepsilon) \cap B(\partial K, \varepsilon^2))$ is not smaller than $\mathrm{Leb}\,(B(0, \varepsilon) \cap B(\tilde{\rho}\theta(x), \tilde{\rho}) \cap \{z \in \mathbb{R}^d : \langle z, \theta(x) \rangle \leqq \varepsilon^2\})$, which does not depend on $x \in \partial K$ and is precisely of order ε^{d+1} as $\varepsilon \downarrow 0$.

The following result is useful for the proof of Corollary 2 as well as Theorem 2.

Lemma 2. *The mapping* $\phi(x) = |x|^{-1}x$ *defines a homeomorphism from* ∂K *onto* $\partial B(0, 1)$ *such that*

$$\rho R^{-2}\,|x - y| \leqq |\phi(x) - \phi(y)| \leqq \rho^{-1}\,|x - y| \qquad \forall x, y \in \partial K.$$

Proof of Lemma 2. Since the mapping under consideration is surjective, it suffices to prove the two inequalities for $x, y \in \partial K$. The second inequality follows from $|x - y|^2 = (|x| - |y|)^2 + |x|\,|y|\,|\phi(x) - \phi(y)|^2 \geqq \rho^2\,|\phi(x) - \phi(y)|^2$. In order to prove the first inequality one may assume without loss of generality that $R = 1$. Further, let $\lambda := |x|/|y| \leqq 1$. Then

$$\begin{aligned} |x - y|^2 - |\phi(x) - \phi(y)|^2 &\leqq |\lambda\phi(x) - \phi(y)|^2 - |\phi(x) - \phi(y)|^2 \\ &= (\lambda - 1)(\lambda + 1 - 2\langle(\phi(x), \phi(y)\rangle) \\ &\leqq 0 \qquad \text{if } \langle \phi(x), \phi(y) \rangle \leqq (\lambda + 1)/2. \end{aligned}$$

Hence one may assume that $\langle \phi(x), \phi(y) \rangle > (\lambda + 1)/2 \geqq \lambda$. The function $\mathbb{R} \ni t \mapsto |x + t(y - x)|^2$ attains its minimum at

$$t_0 := (|x|^2 - \langle x, y \rangle)/|x - y|^2 = (\lambda - \langle \phi(x), \phi(y) \rangle)\,|x|\,|y|/|x - y|^2 < 0.$$

Thus x is a non-trivial convex combination of y and $z := x + t_0(y - x)$. In particular, $|z| \geqq \rho$, because otherwise x would belong to the interior of $\mathrm{conv}\,(B(0, \rho) \cup \{y\})$, which is a subset of $\mathrm{int}\,(K)$. Now the asserted inequality follows from

$$\begin{aligned} |z|^2 &= |x|^2 - (|x|^2 - \langle x, y \rangle)^2/|x - y|^2 \\ &= |x|^2\,|y|^2[|\lambda\phi(x) - \phi(y)|^2 - (\lambda - \langle \phi(x), \phi(y) \rangle)^2]/|x - y|^2 \\ &= |x|^2\,|y|^2(1 + \langle \phi(x), \phi(y) \rangle)(1 - \langle \phi(x), \phi(y) \rangle)/|x - y|^2 \\ &\leqq |\phi(x) - \phi(y)|^2/|x - y|^2. \end{aligned}$$

Proof of Corollary 2. It suffices to show that (4) holds with $C_i = \{X_i\}$ and $\beta = d - 1$. But this is a direct consequence of Lemma 2. For Lemma 2 implies that

$\phi(B(x, \varepsilon) \cap \partial K) \supset B(\phi(x), \rho R^{-2}\varepsilon) \cap \partial B(0, 1)$, and it follows from the definition of H_{d-1} that $H_{d-1}(\phi(A)) \leqq \rho^{1-d} H_{d-1}(A)$ for Borel sets $A \subset \partial K$ (cf. Federer 1969). Hence $H_{d-1}(B(x, \varepsilon) \cap \partial K) \geqq \rho^{d-1} H_{d-1}(B(\phi(x), \varepsilon\rho/R^2) \cap \partial B(0, 1)) \geqq \tilde{\alpha}\varepsilon^{d-1}$ for all $\varepsilon \in [0, R^2/\rho]$ and some constant $\alpha = \alpha(d) > 0$.

Proof of Corollary 3. It suffices to verify (4) with $C_i = \{x \in \partial K : \langle x, U_i \rangle = A_i\}$ and $\beta = 1$. We utilize the following topological fact, which is easily verified: for any $x \in \partial K$ and $\varepsilon \in [0, R/2]$ there exists a continuous path $\gamma : [0, 1] \to \partial K$ such that $\gamma(0) = x$, $|\gamma(\lambda) - x| \leqq \varepsilon$ for all $\lambda \in [0, 1]$ and $|\gamma(1) - x| = \varepsilon$. Then (4) follows from

$$\begin{aligned} \boldsymbol{P}\{\langle z, U_1 \rangle = A_1 \text{ for some } z \in \partial K \cap B(x, \varepsilon)\} &\geqq \boldsymbol{P}\{A_1 \in \{\langle \gamma(\lambda), U_1 \rangle : \lambda \in [0, 1]\}\} \\ &\geqq \boldsymbol{E}|\langle x - \gamma(1), U_1 \rangle|/2R \\ &= \alpha\varepsilon, \end{aligned}$$

where $\alpha := \boldsymbol{E}|\langle s, U_1 \rangle|/(2R) > 0$ for any fixed $s \in \partial B(0, 1)$.

The main tools for proving Theorem 2 are Lemma 2 and

Lemma 3. *Let L be a compact, convex subset of $\mathbb{R}^d$ such that $0 \in \operatorname{int}(L)$. Then* $\rho\rho(L) \leqq d_H(K, L)/d_H(K^*, L^*) \leqq RR(L)$.

Other inequalities relating a set and its polar can be found in Chapter 24.5 of Burago and Zalgaller (1988). The inequalities in Lemma 3 are sharp, as can be seen by setting $L = B(0, \rho)$ or $L = B(0, R)$. Furthermore, a little thought reveals that

$$\liminf_{L \to K} \frac{d_H(K, L)}{d_H(K^*, L^*)} = \rho^2 \quad \text{and} \quad \limsup_{L \to K} \frac{d_H(K, L)}{d_H(K^*, L^*)} = R^2,$$

where convergence is with respect to $d_H(\cdot, \cdot)$.

Proof of Lemma 3. Let us first mention two duality results that are related to (2):

$$h(K^*, s) = g(K, s) := \min\{\lambda > 0 : s \in \lambda K\} \quad \forall s \in \partial B(0, 1); \tag{6}$$

see Schneider (1993, Theorem 1.7.6). The function $g(K, \cdot)$ is the so-called gauge function of K, and $g(K, s)^{-1}$ is the length of the line segment from the origin to ∂K in direction s. Now one can easily derive that

$$R^{-1} = \min_{x \in \partial B(0,1)} g(K, s) = \min_{s \in \partial B(0,1)} h(K^*, s) = \rho(K^*). \tag{7}$$

By compactness and convexity of K and L one may assume that there is a point

$x \in \partial L$ such that $d_H(K, L) = d(x, K)$. (Otherwise interchange K and L.) Then there is an $s \in \partial B(0, 1)$ such that $d(x, K) = h(K, s) + \langle x, s\rangle$, whence

$$\begin{aligned} g(L, \phi(x))^{-1} - g(K, \phi(x))^{-1} &= |x| - \max\{\lambda > 0 : \lambda\phi(x) \in K\} \\ &\geqq |x| - \max\{\lambda > 0 : \lambda\langle\phi(x), s\rangle \leqq h(K, s)\} \\ &= |x| - |x|\,(1 - d_H(K, L)/\langle x, s\rangle) \\ &\geqq d_H(K, L). \end{aligned}$$

Consequently,

$$\begin{aligned} d_H(K, L) &\leqq \|g(L, \cdot)^{-1} - g(K, \cdot)^{-1}\| \\ &= \|g(L, \cdot)^{-1} g(K, \cdot)^{-1}(g(L, \cdot) - g(K, \cdot))\| \\ &\leqq RR(L)\,\|g(L, \cdot) - g(K, \cdot)\| \\ &= RR(L) d_H(K^*, L^*), \end{aligned}$$

by (3) and (6). The other asserted inequality follows by interchanging (K, L) and (K^*, L^*) with the help of (2) and (7).

Proof of Theorem 2. Note first that

$$\begin{aligned} \tilde{K} := \bigcap_{s \in S} H(K^*, s) &= \bigcap_{s \in S} \{x \in \mathbb{R}^d : \langle x, s\rangle \leqq g(K, s)\} \\ &= \bigcap_{z \in Z} \{x \in \mathbb{R}^d : \langle x, z\rangle \leqq 1\} \\ &= Z^* = \operatorname{conv}(Z)^*; \end{aligned}$$

see (6). Generally $\operatorname{conv}(Z)$ is a compact, convex subset of K. If $0 \notin \operatorname{int}(\operatorname{conv}(Z))$, then there exists an $s \in \partial B(0, 1)$ such that $h(Z, s) \leqq 0$. In particular, $d_H(K, \operatorname{conv}(Z)) \geqq d(\rho s, \operatorname{conv}(Z)) \geqq \rho$, and $\{rs : r > 0\} \subset \tilde{K}$, whence $d_H(K^*, \tilde{K}) = \infty$. Therefore one may assume that $0 \in \operatorname{int}(\operatorname{conv}(Z))$. Then Lemma 3, (2) and (7) together yield

$$\begin{aligned} d_H(K^*, \tilde{K}) &\geqq \rho(K^*)\rho(\tilde{K}) d_H(K, \operatorname{conv}(Z)) \geqq R^{-2} d_H(K, \operatorname{conv}(Z)), \\ d_H(K^*, \tilde{K}) &\geqq R(K^*)R(\tilde{K}) d_H(K, \operatorname{conv}(Z)) \\ &= \rho^{-1} d_H(K, \operatorname{conv}(Z))/\rho(\operatorname{conv}(Z)) \\ &\leqq \rho^{-1} d_H(K, \operatorname{conv}(Z))/(\rho - d_H(K, \operatorname{conv}(Z)))^+ \\ &= \rho^{-2} d_H(K, \operatorname{conv}(Z))/(1 - \rho^{-1} d_H(K, \operatorname{conv}(Z)))^+. \end{aligned}$$

The two inequalities for $\pi(\cdot)$ follow straightforwardly from Lemma 2 together with (7).

For the proof of Corollary 4 one needs to know that condition (S) and condition (NF) are equivalent.

Lemma 4. *Condition* (*NF*) *implies condition* (*S*) *with* $l := R/\rho l^*$. *On the other hand, condition* (*S*) *implies condition* (*NF*) *with* $l^* := \rho^3/R^3 l$.

Proof of Lemma 4. Any unit vector in $\mathbb{R}^d$ can be written as $\phi(z)$ as well as $\phi(\tilde{z})$ for some $z \in \partial K, \tilde{z} \in \partial(K^*)$. Let $x,y \in \partial K, \tilde{x},\tilde{y} \in \partial(K^*)$ such that $\phi(\tilde{x}) \in \Theta(K, x)$ and $\phi(\tilde{y}) \in \Theta(K, y)$. This is equivalent to $\phi(x) \in \Theta(K^*, \tilde{x})$ and $\phi(y) \in \Theta(K^*, \tilde{y})$, because

$$\begin{aligned}\phi(x) \in \Theta(K^*, \tilde{x}) &\Leftrightarrow \langle \tilde{x}, \phi(x)\rangle = h(K^*, \phi(x))\\ &\Leftrightarrow \langle \phi(\tilde{x}), \phi(x)\rangle = g(K^*, \phi(\tilde{x}))h(K^*, \phi(x))\\ &\Leftrightarrow \langle \phi(\tilde{x}), \phi(x)\rangle = h(K, \phi(\tilde{x}))g(K, \phi(x))\\ &\Leftrightarrow \phi(\tilde{x}) \in \Theta(K, x);\end{aligned}$$

see (6). Thus if condition (NF) holds, then

$$\begin{aligned}|\phi(\tilde{x}) - \phi(\tilde{y})| &\leqq \rho(K^*)^{-1}\,|\tilde{x} - \tilde{y}|\\ &\leqq (l^*\rho(K^*))^{-1}\,|\phi(x) - \phi(y)|\\ &\leqq (\rho l^*\rho(K^*))^{-1}\,|x - y|\\ &= R(\rho l^*)^{-1}\,|x - y|,\end{aligned}$$

according to Lemma 1 and (7). This implies condition (S) with $l = R/\rho l^*$. Analogously one can show that condition (S) implies condition (NF) with $l^* = \rho^3/R^3 l$.

Proof of Corollary 4. The assertions follow straightforwardly from Theorem 2 and Lemma 1, if we show that

$$\begin{aligned}&\boldsymbol{P}\{U_i \in B(x, \varepsilon)\} \geqq \alpha\varepsilon^{d-1} \qquad \text{and}\\ &\boldsymbol{P}\{\{U_i\}^\perp \cap \partial B(0, 1) \cap B(x, \varepsilon) \neq \emptyset\} \geqq \alpha'\varepsilon \qquad \forall \varepsilon \in [0, 1] \quad \forall x \in \partial B(0, 1)\end{aligned} \tag{8}$$

with positive constants α,α'. The first half of (8) is well-known. As for the second inequality,

$$\begin{aligned}&\boldsymbol{P}\{\{U_i\}^\perp \cap \partial B(0, 1) \cap B(x, \varepsilon) \neq \emptyset\}\\ &\quad\geqq \boldsymbol{P}\{\langle x + z, U_i\rangle = 0 \text{ for some } z \in B(0, \varepsilon) \cap \{x\}^\perp\}\\ &\quad= \boldsymbol{P}\{|\langle x, U_i\rangle| \leqq \varepsilon(1 - \langle x, U_i\rangle^2)^{\frac{1}{2}}\}\\ &\quad= \boldsymbol{P}\{|\langle x, U_i\rangle| \leqq \varepsilon(1 + \varepsilon^2)^{-\frac{1}{2}}\}\\ &\quad\geqq \alpha'\varepsilon \quad \forall \varepsilon \in [0, 1]\end{aligned}$$

with some $\alpha' = \alpha'(d) > 0$.

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